To integrate x to power x

There is no elementary function for finding the integral $\int x^x\,dx$ (at least to me). However, here is a challenge for you to find the definite integral $\int_0^1 x^x\,dx$.

(a) Given $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, express x^x in terms of an infinite series.

(b) Use integration by parts, or otherwise, show that :

$\int x^m (\ln x)^n dx = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$, where ln stands for the logarithm base e and we don't write the integrating constant for simplicity.

(c) **(i)** Use l'Hospital's rule to show that $\lim_{x\to 0^+} x^m (\ln x)^n = 0$

(ii) Show that $\int_0^1 x^n (\ln x)^n dx = -\frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} dx$

And $\int_0^1 x^n (\ln x)^n dx = \frac{(-1)^n n!}{(n+1)^{n+1}}$

(d) Prove that $\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k}$.

(e) Prove that $\int_0^1 x^{-x}\,dx = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^k}$.

(a) $x^x = e^{x \ln x}$

Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we have

$$x^x = 1 + \frac{x \ln x}{1!} + \frac{x^2 (\ln x)^2}{2!} + \frac{x^3 (\ln x)^3}{3!} + \frac{x^4 (\ln x)^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k (\ln x)^k}{k!}$$

(b) **Method 1**

$I = \int x^m (\ln x)^n dx$

Let $u = (\ln x)^n$, $dv = x^m dx$. Then $du = n(\ln x)^{n-1} \frac{1}{x} dx$, $v = \frac{1}{m+1} x^{m+1}$

Using integration by parts, we get:

$$I = \int x^m (\ln x)^n dx = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

Method 2

$$\frac{d}{dx}[x^{m+1} (\ln x)^n] = (m+1) x^m (\ln x)^n + n(\ln x)^{n-1} \frac{1}{x} x^m = (m+1) x^m (\ln x)^n + n(\ln x)^{n-1} x^{m-1}$$

Integrate both sides,

$x^{m+1}(\ln x)^n = (m+1)\int x^m(\ln x)^n dx + n\int x^{m-1}(\ln x)^{n-1}dx$

Therefore, $\int x^m(\ln x)^n dx = \frac{1}{m+1}x^{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m(\ln x)^{n-1}dx$

(c) **(i)** $L = \lim_{x\to 0^+} x^m(\ln x)^n = \lim_{x\to 0^+}\frac{(\ln x)^n}{x^{-m}} =_{(LHR)} \lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln x)^n}{\frac{d}{dx}x^{-m}} = \lim_{x\to 0^+}\frac{n(\ln x)^{n-1}\frac{1}{x}}{(-n)x^{-m-1}}$

$= \lim_{x\to 0^+}\frac{n(\ln x)^{n-1}}{(-n)x^{-m}} =_{(LHR)} \lim_{x\to 0^+}\frac{n(n-1)(\ln x)^{n-2}}{(-n)^2x^{-m}} = \cdots = \lim_{x\to 0^+}\frac{n!}{(-n)^n x^{-m}} = \lim_{x\to 0^+}\frac{n!x^m}{(-n)^n}$

= 0

(ii) By **(b)**, $\int_0^1 x^n(\ln x)^n dx = \frac{1}{n+1}x^{n+1}(\ln x)^n\Big|_0^1 - \frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}dx$

$= 0 - \lim_{x\to 0^+} x^{n+1}(\ln x)^n - \frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}dx$

$= -\frac{n}{n+1}\int_0^1 x^n(\ln x)^{n-1}dx$, by **(c) (i)** where m = n + 1

$= -\frac{n(n-1)}{(n+1)^2}\int_0^1 x^n(\ln x)^{n-2}dx$, by **(b)** and **(c) (i)**

$= \frac{(-1)^n n!}{(n+1)^n}\int_0^1 x^n dx = \frac{(-1)^n n!}{(n+1)^{n+1}}$

(d) $\int_0^1 x^x\, dx = \int_0^1 \sum_{k=0}^{\infty}\frac{x^k(\ln x)^k}{k!}\, dx$, by **(a)**

$= \sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1 x^k(\ln x)^k\, dx$

$= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{(-1)^k k!}{(k+1)^{k+1}}$, by **(c) (ii)**

$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{k+1}} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^k} \approx \mathbf{0.783431}$

(e) **Check:** $x^{-x} = \sum_{k=0}^{\infty}\frac{(-1)^k x^k(\ln x)^k}{k!}$

$\int_0^1 x^{-x}\, dx = \int_0^1 \sum_{k=0}^{\infty}\frac{(-1)^k x^k(\ln x)^k}{k!}\, dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^1 x^k(\ln x)^k\, dx$

$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{(-1)^k k!}{(k+1)^{k+1}} = \sum_{k=0}^{\infty}\frac{(-1)^k 1}{(k+1)^{k+1}} = \sum_{k=1}^{\infty}\frac{1}{k^k} \approx \mathbf{1.29129}$

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